DYNAMIC VIBRATIONS IN HOMOGENEOUS INCOMPRESSIBLE ELASTIC SHELLS

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(*Received November* 1, 1968)

ABSTRACT. The finite Hankel transform is used to find the transient displacement and stresses in thick homogeneous elastic shells subjected to dynamic loads on the surfaces for the following problems : (i) Radial motion of an infinitely long circular cylindrical shell. (ii) radially symmetric motion of a spherical shell. In both the problems the materials have been assumed to be isotropic, but incompressible.

INTRODUCTION

In a recent paper Cinelli (1966) has used finite Hankel transform (Cinelli, 1965) to find the solutions of dynamic vibrations in elastic cylinders and spheres, subjected to dynamic loads on the surfaces. Chakravorty and Chatterjee (1969) has employed Cinelli's direct and concise method to find the stresses and displacements in spherical and cylindrical shells of non-homogeneous isotropic materials, while Chatterjee (1969) has used the same method to the case of radically symmetric motion of a spherical shell of spherically isotropic material. The present paper deals with two problems of vibration of thick shells of incompressible isotropic material, viz., radially symmetric motion of a spherical shell and radial motion of a thick cylindrical shell. The loads applied to both the surfaces of the spherical and cylindrical shells are assumed to be completely arbitrary functions of time, It is proposed to extend the same problems ito the case, where the materials, in addition to being incompressible, are non-homogeneous, in a subsequent paper.

FINITE HANKEL TRANSFORM AND ITS PROPERTIES

The formulae of the new Hankel transform (Cinelli, 1965) are given here for later reference. A bar over small letters indicates the transform variable, where as the prime denotes differentiation, H denoting the integral operator.

$$\bar{f}(\xi_i) = H[f(r)] = \int_a^b r f(r) C_m(r, \xi_i) dr, \qquad a \leqslant r \leqslant b \qquad \ldots (1)$$

$$C_m(r, \xi_i) = \{J_m(\xi_i r)[\xi_i Y'_m(\xi_i a) + h Y_m(\xi_i a)] - Y_m(\xi_i r)[\xi_i J_m'(\xi_i a) + k J_m(\xi_i a)]\} \qquad \ldots (2)$$

$$f(r) = \frac{\pi^2}{2} \sum_{\xi_i} \xi_i^2[\xi_i J'_m(\xi_i b)+kJ_m(\xi_i b)]^2 . \bar{f}(\xi_i) \frac{C_m(T, \xi_i)}{F_m(\xi_i)}, \quad .. \quad (3)$$

$$F_m(\xi_i) = \left\{K^2+\xi_i^2 \left[1-\left(\frac{m}{\xi_i b}\right)^2\right]\right\}[\xi_i J_m(\xi_i a)+hJ_m(\xi_i a)]$$

$$-\left\{h^2+\xi_i^2 \left[1-\left(\frac{m}{\xi_i a}\right)^2\right]\right\} [\xi_i J'_m(\xi_i b)+KJ_m(\xi_i b)], \quad .. \quad (4)$$

where ξ_i is a positive root of

$$[\xi_i Y'_m(\xi_i a)+hY_m(\xi_i a)][\xi_i J'_m(\xi_i b)+kJ_m(\xi_i b)]$$

$$= [\xi_i Y'_m(\xi_i b)+kY_m(\xi_i b)][\xi_i J'_m(\xi_i a)+hJ_m(\xi_i a)] \quad .. \quad (5)$$

$$H\left[\frac{d^2 f}{dr^2}+\frac{1}{r}\frac{df}{dr}-\frac{m^2}{r^2}f\right] = \frac{2}{\pi} . \frac{[\xi_i J'_m(\xi_i a)+hJ_m(\xi_i a)]}{[\xi_i J'_m(\xi_i b)+KJ_m(\xi_i b)]} . [f'(b)+Kf(b)]$$

$$-\frac{2}{\pi}[f'(a)+hf(a)]-\xi^2_i\bar{f}(\xi_i) \quad .. \quad (6)$$

where

J_m, Y_m = Bessel function of the first and second kind, respectively, of order m,
h, k = constant coefficients whose value can be positive, negative or zero,
a, b = inner and outer radii of the shells respectively,
$f(r)$ = arbitrary function of variable r.

RADIAL VIBRATION OF AN INCOMPRESSIBLE ISOTROPIC SPHERICAL SHELL

For the radial vibration of a thick spherical shell, we assume the displacement components in spherical polar coordinates r, θ, ϕ as

$$u_r = u(r, t), \quad u_\theta = u_\varphi = 0 \quad .. \quad (7)$$

The non-vanishing components of strain are

$$e_{rr} = \frac{\partial u}{\partial r} ; \qquad e_{\theta\theta} = e_{\varphi\varphi} = \frac{u}{r} ; \quad .. \quad (8)$$

For compressibility,

$$\text{(i)} \quad \sigma = 0{\cdot}5; \qquad E = 3G ; \quad .. \quad (9)$$

where E, G, σ are young's modulus, shear modulus and poisson's ratio respectively,

$$\text{(ii)} \quad \Delta = e_{rr}+e_{\theta\theta}+e_{\varphi\varphi} = \frac{\partial u}{\partial r}+\frac{2u}{r} = 0 \quad .. \quad (10)$$

The stress-strain relations are given by

$$\left.\begin{aligned} 3G.e_{rr} &= \widehat{rr}-\tfrac{1}{2}(\widehat{\theta\theta}+\widehat{\phi\phi}) = \widehat{rr}-\widehat{\theta\theta}, \\ 3G.e_{\theta\theta} &= \widehat{\theta\theta}-\tfrac{1}{2}(\widehat{\phi\phi}+\widehat{rr}) = \tfrac{1}{2}(\widehat{\theta\theta}-\widehat{rr}), \\ 3G.e_{\varphi\varphi} &= \widehat{\phi\phi}-\tfrac{1}{2}(\widehat{rr}+\widehat{\theta\theta}) = \tfrac{1}{2}(\widehat{\theta\theta}-\widehat{rr}), \end{aligned}\right\} \quad .. \quad (11)$$

so that the components of stress are given by

$$\widehat{rr} = 2G.\frac{\partial u}{\partial r}; \qquad \widehat{\theta\theta} = \widehat{\phi\phi} = 2G.\frac{u}{r} \quad .. \quad (12)$$

The only non-vanishing equation of motion, viz,

$$\frac{\partial}{\partial r}(\widehat{rr})+\frac{2}{r}(\widehat{rr}-\theta\theta) = \rho\,\frac{\partial^2 u}{\partial t^2}, \quad .. \quad (13)$$

with the help of (12) and (10) reduces to

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\,\frac{\partial u}{\partial r}-\frac{4u}{r^2} = \frac{1}{c^2}\cdot\frac{\partial^2 u}{\partial t^2}, \quad .. \quad (14)$$

where

$$c^2 = \frac{2G}{\rho} \quad .. \quad (15)$$

The initial conditions are

$$u = \frac{\partial u}{\partial t} = 0, \qquad \text{at} \quad t = 0, \qquad a \leqslant r \leqslant b \quad .. \quad (16)$$

The boundary conditions are

$$\widehat{rr}(r,t)]_{r=a} = 2G\frac{\partial u}{\partial r} = A(t), \qquad \text{at } r = a, \quad t > 0 \quad .. \quad (17)$$

$$\widehat{rr}(r,t)]_{r=b} = 2G.\frac{\partial u}{\partial r} = B(t) \qquad \text{at } r = b, \quad t > 0 \quad .. \quad (18)$$

Taking $h = k = 0$, and $m = 2$, we get from (6)

$$H\left[\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\,\frac{\partial u}{\partial r}-\frac{4u}{r^2}\right] = \frac{1}{G\pi}\left\{\frac{J_2'(\xi_i a)}{J_2'(\xi_i b)}2Gu'(b)-2Gu'(a)\right\}-\xi_i^2\bar{u}(\xi_i) \quad .. \quad (19)$$

Comparing (17), (18), (19) the appropriate finite Hankel transform for equation (14) is

$$\bar{u} = \bar{u}(\xi_i,t) = \int_a^b r u(r, t) C_2(r, \xi_i) dr, \quad .. \quad (20)$$

where C_2 is given by (2) for $m = 2$.

Applying (20) to (14) and using (17), (18), (19) we have,

$$\frac{1}{c^2} \cdot \frac{\partial^2 \bar{u}}{\partial t^2} = \frac{1}{G\pi} \left\{ \frac{J_2'(\xi_i a)}{J_2'(\xi_i b)} \widehat{rr}(b, t) - \widehat{rr}(a, t) \right\} - \xi_i^2 \bar{u}(\xi_i) \quad .. \quad (21)$$

Rearranging (21), we have, on putting the surface loads.

$$\frac{1}{c_2} \cdot \frac{\partial^2 \bar{u}}{\partial t^2} + \xi_i^2 \bar{u}(\xi_i) = \frac{2}{\pi \rho c^2} \left\{ \frac{J_2'(\xi_i a)}{J_2'(\xi_i b)} B(t) - A(t) \right\} \quad .. \quad (22)$$

Now we use Laplace transform, initial conditions and the convolution integral to get the solution of (22) as

$$\bar{u}(\xi_i,t) = \frac{2}{\pi \rho c \xi_i} \int_0^t \left\{ \frac{J_2'(\xi_i a)}{J_2'(\xi_i b)} B(\tau) - A(\tau) \right\} \operatorname{Sin} \xi c_i (t-\tau) d\tau \quad .. \quad (23)$$

From (3) we have them

$$u(r, t) = \frac{\pi^2}{2} \sum_{\xi_i} \xi_i^2 [\xi_i J_2'(\xi_i b)]^2 . \quad \bar{u}(\xi_i,t) \frac{C_2(r, \xi_i)}{F_2(\xi_i)}, \quad .. \quad (24)$$

where ξ_i and F_2 are given by (5) and (4) respectively with $m = 2$, and $h = k = 0$. Placing (23) into (24) we get the solution as

$$u(r,t) = \frac{\pi}{\sqrt{2G\rho}} \sum_{\xi_i} \xi_i [\xi_i J_2'(\xi_i b)]^2 \frac{C_2(r, \xi_i)}{F_2(\xi_i)} I_1, \quad .. \quad (25)$$

where,

$$I_1 = \int_0^t \left\{ \frac{J_2'(\xi_i a)}{J_2'(\xi_i b)} B(\tau) - A(\tau) \right\} \operatorname{Sin} \xi_i \sqrt{\frac{2G}{\rho}} (t-\tau) d\tau \quad .. \quad (26)$$

The stresses are found as

$$\widehat{rr} = \pi \sqrt{\frac{2G}{\rho}} . S_1 I_1 . \frac{\partial}{\partial r} \{C_2(r, \xi_i)\}, \quad .. \quad (27)$$

$$\widehat{\theta\theta} = \widehat{\phi\phi} = \pi\sqrt{\frac{2G}{\rho}}\, S_1 .\, I_1\, \frac{C_2(r,\xi_i)}{r}, \qquad \ldots (28)$$

where
$$S_1 = \sum_{\xi_i} \frac{\xi_i[\xi_i J'_2(\xi_i b)]^2}{F_2(\xi_i)}, \qquad \ldots (29)$$

and I_1 is given by (26).

RADIAL VIBRATION OF AN INCOMPRESSIBLE HOMOGENEOUS ISOTROPIC CYLINDRICAL SHELL

For the radial vibration of a cylindrical shell we assume the displacement components in cylindrical coordinates r, θ, z as

$$u_r = u(r, t), \qquad u_\theta = u_z = 0. \qquad \ldots (30)$$

The strain components are given by

$$\left.\begin{aligned} e_{rr} &= \frac{\partial u}{\partial r}; \qquad e_{\theta\theta} = \frac{u}{r}; \\ e_{zz} &= e_{r\theta} = e_{rz} = e_{\theta z} = 0 \end{aligned}\right\} \qquad \ldots (31)$$

For incompressibility,

$$\Delta = \frac{\partial u}{\partial r} + \frac{u}{r} = 0 \qquad \ldots (32)$$

and the non-vanishing stress components are given by

$$\widehat{rr} = 2G.\frac{\partial u}{\partial r}; \qquad \widehat{\theta\theta} = 2G.\frac{u}{r} \qquad \ldots (33)$$

The only equation of motion,

$$\frac{\partial}{\partial r}(\widehat{rr}) + \frac{1}{r}(\widehat{rr} - \widehat{\theta\theta}) = \rho\frac{\partial^2 u}{\partial t^2},$$

becomes
$$\frac{\partial^2 u}{ar^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} = \frac{1}{c^2}\cdot\frac{\partial^2 u}{\partial t^2}, \qquad \ldots (34)$$

where c^2 is given by (15).

The initial and the boundary conditions are given by

$$u = \frac{\partial u}{\partial t} = 0, \qquad \text{at } t = 0, \qquad a \leqslant r \leqslant b \qquad \ldots (35)$$

$$\widehat{rr}\,(r, t)]_{r=a} = 2G.\frac{\partial u}{\partial r} = A(t), \text{ at } r = a, \quad t > 0 \qquad .. \quad (36)$$

$$\widehat{rr}\,(r, t)]_{r=b} = 2G.\frac{\partial u}{\partial r} = B(t), \text{ at } r = b, \quad t > 0 \qquad .. \quad (37)$$

Taking $h = k = 0$ and $m = 1$, we get from (6)

$$H\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right] = \frac{1}{\pi G}\left\{\frac{J_1'(\xi_i a)}{J_1'(\xi_i b)} 2G.\,u'(b) - 2Gu'(a)\right\} - \xi_i^2 \bar{u}(\xi_i) \qquad .. \quad (38)$$

Comparing (36), (37) and (38), we see that the appropriate finite Hankel transform for equation (34) is

$$\bar{u} = \bar{u}(\xi_i, t) = \int_a^b r u(r, t) C_1(r, \xi_i) dr \qquad .. \quad (39)$$

Applying (39) to equation (34) and using (36), (33), (38), we have

$$\frac{1}{c^2}.\frac{\partial^2 \bar{u}}{\partial t^2} + \xi^2 \bar{u}(\xi_i) = \frac{2}{\pi \rho c^2}\left\{\frac{J_1'(\xi_i a)}{J_1'(\xi_i b)}.\,B(t) - A(t)\right\} \qquad .. \quad (40)$$

Now we proceed exactly in the same manner as in the previous case, and the displacement is found to be

$$u(r, t) = \frac{\pi}{\sqrt{2G\rho}} \sum_{\xi_i} \xi_i \frac{[\xi_i J_1'(\xi\, b)]^2}{F_1(\xi_i)} C_0(r, \xi_i) I_2, \qquad .. \quad (41)$$

where

$$I_2 = \int_0^t \left\{\frac{J_1'(\xi_i a)}{J_1'(\xi_i b)}.\,B(\tau) - A(\tau)\right\} \operatorname{Sin} \xi_i \sqrt{\frac{2G}{\rho}}\,(t-\tau) d\tau \qquad .. \quad (42)$$

The stresses are given by

$$\widehat{rr} = \pi\sqrt{\frac{2G}{\rho}}.\,S_2 I_2.\,\frac{\partial}{\partial r}\{C_1(r, \xi_i)\}, \qquad .. \quad (43)$$

$$\widehat{\theta\theta} = \pi\sqrt{\frac{2G}{\rho}}.\,S_2 I_2.\,\frac{C_1(r, \xi_i)}{r} \qquad .. \quad (44)$$

where

$$S_2 = \sum_{\xi_i} \xi_i \frac{[\xi_i J_1'(\xi_i b)]^2}{F_1(\xi_i)} \qquad .. \quad (45)$$

Finally, the author thanks Dr. J. G. Chakraborty, Department of Applied Mathematics, University of Calcutta for his kind guidance in preparing this paper.

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